

Ultrarelativistic N -boson systems

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Abstract. General analytic energy bounds are derived for N -boson systems governed by ultrarelativistic Hamiltonians of the form

$$H = \sum_{i=1}^N \|\mathbf{p}_i\| + \sum_{1 \leq i < j}^N V(r_{ij}),$$

where $V(r)$ is a static attractive pair potential. It is proved that a translation-invariant model Hamiltonian H_c provides a lower bound to H for all $N \geq 2$. This result was conjectured in an earlier paper but proved only for $N = 2, 3, 4$. As an example, the energy in the case of the linear potential $V(r) = r$ is determined with error less than 0.55% for all $N \geq 2$.

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1. Introduction

We consider first the semirelativistic N -body Hamiltonian H given by

$$H = \sum_{i=1}^N \sqrt{\|\mathbf{p}_i\|^2 + m^2} + \sum_{1 \leq i < j}^N V(r_{ij}), \quad (1)$$

and the following model Hamiltonian H_c

$$H_c = \sum_{1 \leq i < j}^N \left[\gamma^{-1} \sqrt{\gamma \|\mathbf{p}_i - \mathbf{p}_j\|^2 + (mN)^2} + V(r_{ij}) \right], \quad (2)$$

where $\gamma = \binom{N}{2} = \frac{1}{2}N(N-1)$. If $\Psi(\rho_2, \rho_3, \dots, \rho_N)$ is the lowest boson eigenstate of H expressed in terms of Jacobi relative coordinates, then it was proved in Ref. [1] that the model facilitates a ‘reduction’ $\langle H_c \rangle = \langle \mathcal{H} \rangle$ to the expectation of a one-body Hamiltonian \mathcal{H} given by

$$\mathcal{H} = N \sqrt{\lambda p^2 + m^2} + \gamma V(r), \quad \lambda = \frac{2(N-1)}{N}. \quad (3)$$

The question remains as to the relation between H and the model H_c . It is known from earlier work (discussed in [1]) that the **lower bound conjecture**

$$\langle H \rangle \geq \langle H_c \rangle \quad (4)$$

is true for the following cases: for the harmonic oscillator $V(r) = vr^2$, for all attractive $V(r)$ in the nonrelativistic large- m limit, and for static gravity $V(r) = -v/r$. This list was augmented in Ref. [1] by the following cases: in general for $N = 3$, and, if $m = 0$, for $N = 4$. The purpose of the present article is to extend this list to include the ultrarelativistic cases $m = 0$ for all $N \geq 2$ and arbitrary attractive $V(r)$.

2. The general lower bound for $m = 0$.

It was shown in Ref. [1] that the non-negativity of the expectation $\langle \delta(m, N) \rangle$ is sufficient to establish the validity of the conjecture (4), where

$$\delta(m, N) = \sum_{i=1}^N \sqrt{\|\mathbf{p}_i\|^2 + m^2} - \frac{2}{N-1} \sum_{1 \leq i < j}^N \sqrt{\frac{N-1}{2N} \|\mathbf{p}_i - \mathbf{p}_j\|^2 + m^2}. \quad (5)$$

Thus for the new cases we are now able to treat we must consider $\langle \delta(0, N) \rangle$. By using the necessary boson permutation symmetry of Ψ , the expectation value we need to study is reduced to

$$\langle \delta(0, N) \rangle = N \left\langle \|\mathbf{p}_1\| - \sqrt{\frac{N-1}{2N}} \|\mathbf{p}_1 - \mathbf{p}_2\| \right\rangle. \quad (6)$$

The principal result of this paper, the lower bound for $m = 0$ and all $N \geq 2$, is an immediate consequence of the following:

Theorem 1 $\langle \delta(0, N) \rangle = 0$.

Proof of Theorem 1

Without loss of generality we adopt in momentum space a coordinate origin such that $\sum_{i=1}^N \mathbf{p}_i := \mathbf{p} = \mathbf{0}$. We define the mean lengths

$$\langle \|\mathbf{p}_1\| \rangle := k \quad \text{and} \quad \langle \|\mathbf{p}_1 - \mathbf{p}_2\| \rangle := d. \quad (7)$$

We wish to make a correspondence between mean lengths such as k and d and the sides of triangles that can be constructed with these lengths. We consider the triangle formed by the three vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2\}$. We suppose that the corresponding angles in this triangle are $\{\phi_{12}, \theta_1, \theta_2\}$ (the same notation is used for other similar triples). We now consider projections of one side on a unit vector along an adjacent side and define the mean angles ϕ and θ by the relations

$$\langle \|\mathbf{p}_1\| \cos(\phi_{12}) \rangle := \langle \|\mathbf{p}_1\| \rangle \cos(\phi)$$

and

$$\langle \|\mathbf{p}_1 - \mathbf{p}_2\| \cos(\theta_1) \rangle := \langle \|\mathbf{p}_1 - \mathbf{p}_2\| \rangle \cos(\theta).$$

Thus, on the average, this triangle is isosceles with one angle ϕ and the other two angles θ . Since $\mathbf{p} = \mathbf{0}$, we have $\langle \mathbf{p}_1 \cdot \mathbf{p} \rangle = 0$. Hence

$$\|\mathbf{p}_1\|^2 + \sum_{i=2}^N \|\mathbf{p}_1\| \|\mathbf{p}_i\| \cos(\phi_{1i}) = 0.$$

Thus, by dividing by $\|\mathbf{p}_1\|$ and using boson symmetry, we find

$$\langle (\|\mathbf{p}_1\| + (N-1)\|\mathbf{p}_2\| \cos(\phi_{12})) \rangle = \langle \|\mathbf{p}_1\| (1 + (N-1) \cos(\phi_{12})) \rangle = 0.$$

We therefore conclude that $k(1 + (N-1) \cos(\phi)) = 0$, that is to say

$$\cos(\phi) = -\frac{1}{N-1}.$$

We now consider again the triangle formed by the three vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2\}$. We have immediately from the dot product $\mathbf{p}_1 \cdot (\mathbf{p}_1 - \mathbf{p}_2)$

$$\|\mathbf{p}_1\| \|\mathbf{p}_1 - \mathbf{p}_2\| \cos(\theta_1) = \|\mathbf{p}_1\| (\|\mathbf{p}_1\| - \|\mathbf{p}_2\| \cos(\phi_{12})).$$

By dividing by $\|\mathbf{p}_1\|$ and taking means we obtain

$$d \cos(\theta) = k(1 - \cos(\phi)).$$

But $\theta = (\pi/2 - \phi/2)$ and $\cos(\phi) = -1/(N-1)$. Hence we conclude

$$\frac{k}{d} = \left(\frac{N-1}{2N} \right)^{\frac{1}{2}}.$$

This equality establishes Theorem 1. □

3. The linear potential

We apply the new bound to the case of the linear potential $V(r) = r$. The weaker $N/2$ lower bound (discussed in Ref. [1]) is always available, but, up to now, we knew no way of obtaining tight bounds for this problem. For a comparison upper bound, we use a Gaussian trial function Φ and the original Hamiltonian H to obtain a scale-optimized variational upper bound $E \leq E_g = (\Phi, H\Phi)$. As we showed in Ref. [1], for the linear potential $V(r) = r$ in three spatial dimensions, the conjecture (now proven) implies that the N -body bounds are given for $N \geq 2$ by

$$N \left(\frac{(N-1)^3}{2N} \right)^{\frac{1}{4}} e = E_c^L \leq E \leq E_g^U = 4N \left(\frac{(N-1)^3}{2N\pi^2} \right)^{\frac{1}{4}}, \quad (8)$$

where $e \approx 2.2322$ is the bottom of the spectrum [2] of the one-body problem $h = \|\mathbf{p}\| + r$. From (8) we see that the ratio $R = E_g/E_c = 4/(\pi^{\frac{1}{2}}e) \approx 1.011$. The energy of the ultrarelativistic many-body system with linear pair potentials is therefore determined by these inequalities with error less than 0.55% for all $N \geq 2$. Earlier we were able to obtain such close bounds for all N only for the harmonic oscillator [3].

4. Conclusion

We have enlarged the number of semirelativistic problems that satisfy the lower-bound conjecture $\langle H \rangle \geq \langle H_c \rangle$ to include all problems with $m = 0$ and $N \geq 2$. An extension of the geometric reasoning used in Ref. [1] from pyramids to more general simplices would perhaps have provided an alternative proof. However, the more algebraic approach adopted here, relying in the end on mean angles in a triangle, seemed to provide a more independent and robust approach.

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